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Wigner function description of a.c. transport through a two-dimensional quantum point contact

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Abstract. We have calculated the admittance of a two-dimensional quantum point contact (QPC) using a novel variant of the Wigner distribution function (WDF) formalism. In the semiclassical approximation, a Boltzmann-like equation is derived for the *partial WDF* describing both propagating and non-propagating electron modes in an effective potential generated by the adiabatic QPC. We show that this quantum kinetic approach leads to the well known stepwise behaviour of the real part of the admittance (the conductance), and of the imaginary part of the admittance (the emittance), in agreement with the latest results derived by Christen and Büttiker, which is determined by the number of propagating electron modes.

It is shown that the emittance is sensitive to the geometry of the QPC, and can be controlled by the gate voltage. We have established that the emittance has contributions corresponding to both quantum inductance and quantum capacitance. Stepwise oscillations in the quantum inductance are determined by the harmonic mean of the velocities for the propagating modes, whereas the quantum capacitance is a significant mesoscopic manifestation of the non-propagating (reflecting) modes.

1. Introduction

Recent technological progress in manufacturing small-scale solid-state structures has made possible the fabrication of devices involving two-dimensional electronic systems (2DES) in the quantum ballistic regime. One particular system that has attracted considerable attention is the quantum point contact (QPC) (see, e.g. [1–21]), which is fabricated by putting a split gate on top of a GaAs–AlGaAs heterostructure, thereby creating a narrow constriction in a two-dimensional electron gas (2DEG). Since in the ballistic regime the electrons do not experience any collisions, propagation through the point contact is analogous to propagation of the electromagnetic wave through a waveguide. The width of the QPC, which is controlled by the gate voltage, can be of the same order of magnitude as the Fermi wavelength and it governs the number of modes that can propagate through the constriction.

In QPC systems several experimental investigations [2–8] have demonstrated quantum coherent phenomena, including quantization of the d.c. conductance versus the gate voltage (or the number of propagating modes through the QPC). The theory of this phenomenon

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[1, 9–11] explains the d.c. conductance quantization as a consequence of adiabatic transit of an electron wave through the QPC with smooth boundaries. In an adiabatic geometry (see figure 1), which is smooth on the scale of the Fermi wavelength, the longitudinal and transverse motion of electrons can be (approximately) separated in the Schrödinger equation [1, 10]. In this case the number of transverse quantization modes is an adiabatic invariant, and the transverse energy plays the role of potential energy for the one-dimensional (1D) longitudinal motion of each mode. Depending on whether the total energy of a given electron state is greater or less than the effective potential energy of a given mode, the mode is propagating or non-propagating (see figure 2).

To date both experimental and theoretical studies of QPCs have been devoted mainly to investigations of the d.c. conductance and the d.c. transport. It is clear, however, that the investigation of the *a.c. transport* can provide additional information, since a finite frequency introduces a new time scale and may reveal qualitatively new effects, particularly if the new time scale is of the order of other characteristic times of the system. The a.c. conductance has been considered by Büttiker *et al* [14–21], who established that the a.c. transport is described by the a.c. admittance $Y = 1/Z = G - \mathrm{i}\omega\mathcal{E}$ at frequency ω, where Z is the impedance. The real part of the a.c. admittance, G, is the conductance, and the imaginary part of Y, which is proportional to $\mathcal{E}$, was first introduced by Büttiker [17] as the *emittance*. In papers [14–21], the general expressions for the electrochemical capacitance and for the displacement current were derived, and the step-like behaviour of the QPC emittance in synchronism with the conductance steps, was established. Christen and Büttiker [19] also discussed the low-frequency QPC emittance of the quantized Hall conductors, and in [20] the authors used the scattering approach for the investigation of the nonlinear current–voltage characteristic of mesoscopic conductors. In papers [14–21], the emittance was expressed in terms of the geometric capacitance, transmission probability, and the densities of states of the 'mesoscopic capacitor plates' [18]. The approach used in [18] takes the Coulomb interaction into account self-consistently by considering two contributions to the displacement charge—one part which neglects screening is determined by the kinetic contribution, a second part corresponds to a screening charge which is due to the shifts of the band bottoms.

In [22] the authors considered the frequency dependence of admittance for a two-dimensional quantum wire, using a current conservation formalism. It was shown in [22] that the correction to the d.c. conductance (in the low-frequency regime) due to a time-dependent potential was related to the local partial density of states, and this correction was calculated numerically. The authors of [23–26] considered the a.c. kinetic response of the resonant tunnel junction (see also [27] where the time-dependent transport through resonant tunnelling systems was considered). In particular, in [23] (see also references therein) the Wigner distribution function approach was used to calculate the a.c. transport through the time-periodic quantum-well resonant-tunnelling diode. The results derived in [23] were based on the 'numerically exact' calculations of the time evolution of the Wigner distribution function, and therefore were characteristically agreeable with the results based on the corresponding time-dependent Schrödinger equation.

In the present paper we extend the studies of the response of a QPC to an a.c. field by developing a simple method, based on the Wigner distribution function (WDF) formalism [28, 29], for calculating the transport characteristics. Our approach allows us to represent the emittance in terms of the capacitance and the inductance, which are expressed in an explicit form through the microscopic characteristics. The effectiveness of the WDF approach to the modelling of small mesoscopic devices was demonstrated in [30, 31]. In section 2, using the assumption of adiabaticity, we derive a Boltzmann-like quantum kinetic equation for a

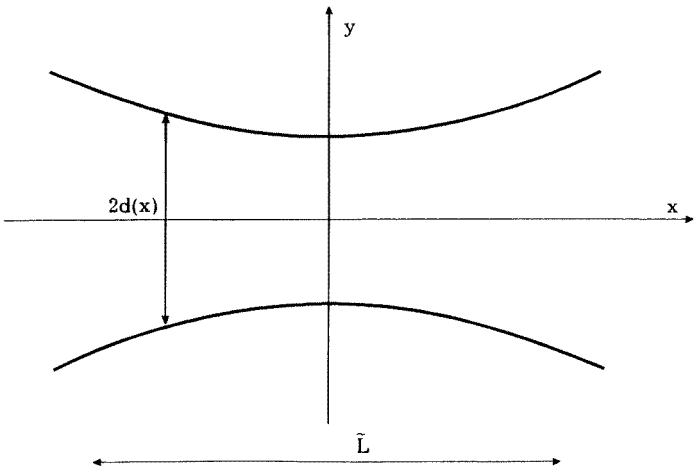

Figure 1. The geometry of the quantum point contact. The width is denoted by $2d(x)$, the narrowest width is $2d_0$, and the effective length is $2\tilde{L}$.

partial WDF describing transport in the quantum ballistic constriction. This equation allows us to treat the 2DES in a QPC in terms of classical trajectories for the effective 1D motion. We assume that the local self-consistant electric field in the QPC is known. Generally, this field can be found from the Poisson equation. However, in the low-frequency approximation considered in this paper (when the characteristic time period of the electric field is much smaller then the characterestic time of the electron's flow through the QPC), only the voltage determines the a.c. current through the QPC. In this approximation only the quantum capacitance depends on the shape of the electric field in the QPC. This approximation is well known and widely used. At the same time, this simplification allows us to concentrate our attention on the peculiarities connected with the developed approach, and on the new effects such as the contribution of inductance into the emittance, and explicit expressions for the parameters and characteristics of the system.

In section 3 we demonstrate how the a.c. admittance of the QPC can be calculated from the propagating and non-propagating (reflected) electron modes. Our approach recovers the quantized behaviour as a function of gate voltage of the real part of the admittance (the conductance), consistent with previous calculations [1, 10] using the Landauer formula [32]. Our approach also allows us to demonstrate that the emittance $\mathcal{E}$ has a negative part, which is a quantum inductance, to which all the propagating electron modes contribute and whose value is determined by the harmonic mean of the electron velocities in the quantized electron modes. The non-propagating electron modes determine the positive contribution to the emittance $\mathcal{E}$, which is a quantum capacitance, and which depends on a geometrical form of the QPC as controlled by the gate voltage. The conclusions are outlined in section 4.

2. Kinetic equations in a quantum ballistic constriction

To find the conductivity of a 2DES in a QPC form (see figure 1), taking into account both frequency dependence and spatial dispersion, we will apply the approach based on the Wigner distribution function [28, 29],

$$f_p^W(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{r}' \,\mathrm{Tr}\left\{\hat{\rho}\exp\left[-\frac{\mathrm{i}}{\hbar}\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}(\boldsymbol{r})\right)\boldsymbol{r}'\right]\Psi^+(\boldsymbol{r}-\boldsymbol{r}'/2)\Psi(\boldsymbol{r}+\boldsymbol{r}'/2)\right\}. \tag{1}$$

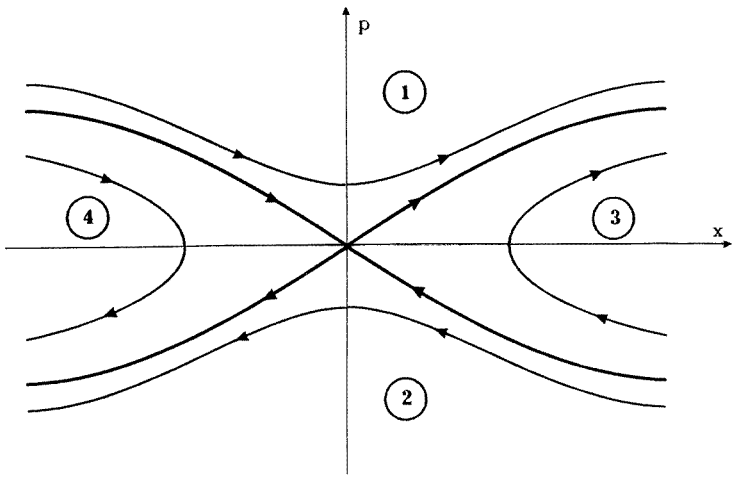

Figure 2. The plane of phase trajectories for one-dimensional motion determined by the conservation of the integrals of motion. The heavy lines are separatrices that separate the propagating modes (regions 1 and 2) and non-propagating (reflecting) modes (regions 3 and 4).

Here $\hat{\rho}$ is the statistical operator of the system; $\Psi^{+}(\boldsymbol{r})$ and $\Psi(\boldsymbol{r})$ are, respectively, the Fermi operators of the creation and annihilation of particles at the point $\boldsymbol{r}$, and $\boldsymbol{A}$ is the vector potential of the electromagnetic field. When the characteristic scale of the spatial inhomogeneity exceeds both the radius of interaction among the particles and the electron's de Broglie wavelength, the kinetic equation for the WDF (1) assumes a form equivalent to the classical kinetic equation [29],

$$\frac{\partial f_p^W}{\partial t} + \boldsymbol{v}\frac{\partial f_p^W}{\partial \boldsymbol{r}} + e\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}, \boldsymbol{B}]\right\}\frac{\partial f_p^W}{\partial \boldsymbol{p}} = \hat{I}\{f_p^W\} \tag{2}$$

where as usual $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, and e is the charge and $\boldsymbol{v}$ the velocity of conduction electrons. Equation (2) is valid for the extended (in the x–y-plane) 2DES, when the typical scales of the inhomogeneity (k^{-1}, d) are much smaller than the characteristic distance between the particles $n^{-1/2}$: $k, 1/d \ll n^{-1/2}$, where k^{-1} is the wavelength of the electromagnetic field, d is a characteristic geometrical scale of the system, and n is the density of the 2DEG. The characteristic distance between the particles is $\sim n^{-1/2}$ due to the weak screening in the 2DEG.

The collision integral, $\hat{I}\left\{f_p^W\right\}$, in equation (2), differs essentially from the classical collision integral, since the quantum transitions included in $\hat{I}\left\{f_p^W\right\}$ reflect the character of the particle statistics and the distinction of the WDF from the classical one [29]. The equilibrium WDF sets the collision integral $\hat{I}\left\{f_p^W\right\}$ to zero.

Using the definition of the WDF, we can express the charge density ρ and the current density $\boldsymbol{j}$, respectively, as [28, 29],

$$\rho(t, \boldsymbol{r}) = \frac{2e}{(2\pi\hbar)^2}\int \mathrm{d}^2\boldsymbol{p}\, f_p^W(\boldsymbol{r}) \tag{3}$$

$$\boldsymbol{j}(t, \boldsymbol{r}) = \frac{2e}{(2\pi\hbar)^2}\int \mathrm{d}^2\boldsymbol{p}\, \boldsymbol{v} f_p^W(\boldsymbol{r}). \tag{4}$$

Despite the evident analogy with the classical distribution function, it is well known that the WDF does not have an interpretation as the probability density, since it can take

both positive and negative values, but the integrated values shown in equations (3) and (4) have the usual physical meanings.

For a finite system, when a 2DEG is located in a bounded region (see figure 1) characterized by distances d of the order of the Fermi wavelength, the left-hand side of the kinetic equation (2) changes its form. Using a standard procedure [28, 29], one can obtain the kinetic equation for the WDF in the 2DES within the strip-like restricted region $|y| < d(x)$, $d(x) =$ constant, which can be written in the form

$$\frac{\partial f_{\boldsymbol{p}}^{W}}{\partial t} + \boldsymbol{v}\frac{\partial f_{\boldsymbol{p}}^{W}}{\partial \boldsymbol{r}} + e\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}, \boldsymbol{B}]\right\}\frac{\partial f_{\boldsymbol{p}}^{W}}{\partial \boldsymbol{p}} + \frac{4\mathrm{sgn}(y)}{m\pi\hbar} \times \int_{-\infty}^{\infty} \mathrm{d}p_y'\, p_y' \cos\left[\frac{2(p_y - p_y')}{\hbar}(d - |y|)\right] f^{W}_{p_x, p_y'} = \hat{I}\{f_{\boldsymbol{p}}^{W}\} \tag{5}$$

where sgn(y) is the sign function. The integral term on the left-hand side of equation (5) arises from the transverse quantization. The presence of this term precludes the naive application of the classical treatment, based on the trajectories, for the solution of the kinetic equation (5) for the WDF. Note that the Boltzmann-like equation (2) can be used in the infinite system when the characteristic size of the inhomogeneities due to the electromagnetic field significantly exceeds the electron de Broglie wavelength. In this case, the integral term in equation (5) vanishes, and (5) transforms into (2).

If $d(x) \neq$ constant, the kinetic equation for the WDF assumes an even more complicated form. To overcome these difficulties, we will invoke the *adiabatical* assumption [1] for the structure of the QPC shown in figure 1. Explicitly, we shall assume that the constriction is sufficiently long and smooth, the criterion

$$d'(x) \simeq d(x)/\tilde{L} \ll 1$$

is met (where $2\tilde{L}$ is the length of the constriction), and that the transport is adiabatic. With this assumption, which has been discussed and analysed in [1, 10], the variables in the Schrödinger equation can be separated, and the eigen-wavefunction can be written in the form,

$$\psi_n(x, y) = \psi_n(x)\Phi_n[y, d(x)] \tag{6}$$

where the transverse wavefunction

$$\Phi_n(y) = \frac{1}{\sqrt{d(x)}} \sin\left\{\frac{\pi n[y + d(x)]}{2d(x)}\right\} \theta[d^2(x) - y^2] \tag{7}$$

should satisfy the boundary conditions

$$\Phi_n(y)|_{y=\pm d(x)} = 0 \tag{8}$$

and $\theta(x)$ is the Heaviside single-step function.

One can then derive an effective Hamiltonian for the longitudinal wavefunction $\psi_n(x)$ as

$$\hat{H} = -\frac{\hbar^2}{2m}\partial_{xx}^2 + \varepsilon_n(x) + e\phi(x). \tag{9}$$

In (9) $\phi(x, y)$ is an electric potential, and $\phi(x)$ is the averaged electric potential with respect to the transverse coordinate y,

$$\phi(x) = \frac{1}{2d(x)} \int_{-d}^{d} \mathrm{d}y\, \phi(x, y).$$

The electric potential $\phi(x, y)$ is assumed to be a smoothly varying function of the transverse coordinate y within the constriction region $|y| < d(x)$. Due to the transverse quantization, the energy of the transverse motion $\varepsilon_n(x)$ in the Hamiltonian (9) has the form

$$\varepsilon_n(x) = \frac{\pi^2 n^2 \hbar^2}{8md^2(x)}. \tag{10}$$

With our assumptions, the transverse quantum number n is an adiabatic integral of motion. Hence we can consider the motion of electrons in the QPC as for a set of effective 1D electron systems enumerated by n. Each effective electron system is located in both the potential $\varepsilon_n(x)$ and the self-consistent electrical potential $\phi(x)$. We can introduce the *partial WDF* (PWDF) as

$$f_n^W(x, p_x) = \int \mathrm{d}x' \exp\left(-\frac{\mathrm{i}p_x x'}{\hbar}\right) \mathrm{Tr}\,\hat{\rho}\Psi_n^+(x - x'/2)\Psi_n(x + x'/2). \tag{11}$$

Using (11), we can represent the WDF in the form

$$f_{\boldsymbol{p}}^W(\boldsymbol{r}) = \sum_{n=1}^{\infty} f_n^W(x, p_x) \int_{-\infty}^{\infty} \mathrm{d}y' \exp\left(-\frac{\mathrm{i}p_y y'}{\hbar}\right) \Phi_{n,x}(y - y'/2)\Phi_{n,x}(y + y'/2). \tag{12}$$

We can derive the equation for the PWDF (11) with the use of the Wigner transformation [28, 29]:

$$\frac{\partial f_n^W}{\partial t} + v_x \frac{\partial f_n^W}{\partial x} + \left[-\frac{\partial \varepsilon_n(x)}{\partial x} + eE(x)\right] \frac{\partial f_n^W}{\partial p}(x, p) = \hat{I}\{f_{\boldsymbol{p}}^W\} \tag{13}$$

where $p \equiv p_x$ and

$$E(x) = -\frac{\partial \phi(x)}{\partial x}.$$

In terms of the PWDF the non-equilibrium charge density and current density can be defined as

$$\rho(x, y) = \sum_{n=1}^{\infty} \rho_n(x)\Phi_n^2(y) \tag{14}$$

$$j(x, y) = \sum_{n=1}^{\infty} j_n(x)\Phi_n^2(y) \tag{15}$$

where $\rho_n(x)$ and $j_n(x)$ are the partial charge and current densities:

$$\rho_n(x) = \frac{e}{\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}p[f_n^W(x, p) - f_n^{W(0)}(x, p)] \tag{16}$$

$$j_n(x) = \frac{e}{\pi\hbar m} \int_{-\infty}^{\infty} \mathrm{d}p\, p f_n^W(x, p). \tag{17}$$

In (16), $f_n^{W(0)}(x, p)$ is the equilibrium PWDF.

The motivation for introducing the PWDF is now clear. In contrast to equation (6), the kinetic equation (13) describing for the PWDF *does* have the form of a classical kinetic equation in the presence of an effective potential $\varepsilon_n(x)$. Hence the solution of this equation *can* be described by the characteristics, i.e. by the classical trajectories.

The formalism used in this paper is based on the assumption that the kinetic equation for the WDF can also involve the collision integral. It is well known (see, e.g. [29]) that this can be realized as crystal periodicity violation, which is a source of electron scattering which does not distort (or distort weakly) the electron spectrum of the ideal crystal. In

this way a weak disorder can be taken into account within the WDF formalism. Certainly, the impurity scattering in a form of the collision integral for the WDF must be treated self-consistently using, for example, the self-consistent Born approximation, which is the simplest method that is free from divergences. In other words, the collision integral can be described in terms of the relaxation frequency depending on the electron energy. It is clear, that in the case when the current carriers have a high mobility, and if the frequencies of the electromagnetic field are sufficiently high, the approximation for the collision integral is justified. The forms of the electron–phonon and electron–impurity collision integrals are too complicated [29]. However, we shall consider here the effects associated with the linear response to the electric field. In this case, the WDF can be found in a linear approximation with respect to the external electric field $\boldsymbol{E}$. It is well known [29] that for a description of the high-frequency effects ($\omega \gg \nu$) in a sample with a high electron mobility, the collision integral can be treated in terms of the momentum relaxation frequency ν, while the mean free path time is $1/\nu$.

In other words, the collision integral in (13) includes quantum transitions [29] and intermixing of the different electron modes (the different PWDF). In the following, we assume a quasi-ballistic regime of transport through the QPC, and we will approximate the collision integral by a single momentum relaxation frequency,

$$\hat{I}_n\{f_{\boldsymbol{p}}^W\} = -\nu[f_n^W(x, p) - f_n^{W(0)}] \tag{18}$$

where $f_n^{W(0)}$ is the equilibrium PWDF. The equilibrium distribution function $f_n^{W(0)}$ within the adiabatic approximation is given by

$$f_n^{W(0)}(x, p) = n_F\left\{\frac{p^2/2m + \varepsilon_n(x) - \mu}{T}\right\} \qquad n_F(x) = (1 + \mathrm{e}^x)^{-1}. \tag{19}$$

The function $n_F(x)$ is the Fermi function with the effective chemical potential $\mu - \varepsilon_n(x)$, where μ is the equilibrium chemical potential of the 2DEG. The effective chemical potential varies smoothly as a function of the longitudinal coordinate x. In this paper we are interested in the linear response, so we expand the PWDF about its equilibrium form

$$f_n^W(x, p) = f_n^{W(0)}(x, p) + f_n(x, p). \tag{20}$$

The kinetic equation linearized in the electric field, $E(x, t) = E(x)\exp(-\mathrm{i}\omega t)$, becomes

$$\frac{p}{m}\frac{\partial f_n}{\partial x} - \frac{\partial \varepsilon_n(x)}{\partial x}\frac{\partial f_n}{\partial p} + (\nu - \mathrm{i}\omega) f_n = -eE\frac{\partial f_n^{W(0)}}{\partial p}. \tag{21}$$

The natural method for solving the kinetic equation (21) is the method of characteristics. The characteristics of this equation are the phase trajectories of a 1D motion in the potential $\varepsilon_n(x)$, which is determined from the integral of motion, *viz* the total energy ε:

$$\varepsilon = \frac{p^2}{2m} + \varepsilon_n(x) = \text{constant} \tag{22}$$

We will consider a reflection symmetric QPC, i.e. $d(x) = d(-x)$. For this case the phase portrait is shown in figure 2. The heavy lines in figure 2 denote the *separatrices*, which pass through the hyperbolic point $p = 0$, $x = 0$ and separate the phase space into four regions, within which four sets of phase trajectories exist.

The regions of propagating trajectories ($\varepsilon > \varepsilon_n(0)$) occupy the regions (see figure 2):

(1) $\varepsilon > \varepsilon_n(0), p > 0$ and (2) $\varepsilon > \varepsilon_n(0), p < 0$.

The regions of non-propagating (reflecting) trajectories ($\varepsilon < \varepsilon_n(0)$) occupy:

(3) $\varepsilon < \varepsilon_n(0), x > 0$ and (4) $\varepsilon < \varepsilon_n(0), x < 0$.

Within each region, one can find the solution of the kinetic equation for the PWDF and derive the general formula for the partial charge ρ_n and the current densities j_n. Here we consider the most interesting case, when the temperature is very low ($T \to 0$, $T \ll \mu$), so that we have a clear separation between propagating ($\varepsilon_n(0) < \mu$) and reflecting ($\varepsilon_n(0) > \mu$) channels.

For the 'open' (i.e. propagating) channels

$$\rho_n(x) = \frac{2e^2}{h}\frac{1}{v_n(x)}\int_{-L}^{L} \mathrm{d}x'\, E(x')\mathrm{sgn}(x-x')\exp[\mathrm{i}\omega^*\tau_n(x,x')\mathrm{sgn}(x-x')] \tag{23}$$

$$j_n(x) = \frac{2e^2}{h}\int_{-L}^{L} \mathrm{d}x'\, E(x')\exp[\mathrm{i}\omega^*\tau_n(x,x')\mathrm{sgn}(x-x')] \tag{24}$$

where $\omega^* = \omega + \mathrm{i}\nu$, $v_n(x) = \sqrt{(2/m)[\mu - \varepsilon_n(x)]}$, and

$$\tau_n(x,x') = \int_{x'}^{x} \frac{\mathrm{d}x''}{v_n(x'')}. \tag{25}$$

For the closed channels (reflecting modes)

$$\rho_n(x) = \frac{2e^2}{h}\frac{\mathrm{sgn}(x)}{v_n(x)}\int_{x_n}^{L} \mathrm{d}x' E(x'\mathrm{sgn}(x))\{\mathrm{sgn}(|x|-x')\exp[\mathrm{i}\omega^*\tau_n(|x|,x') \times \mathrm{sgn}(|x|-x')] - \exp[\mathrm{i}\omega^*(\tau_n(|x|,x_n)+\tau_n(x',x_n))]\}\theta(|x|-x_n) \tag{26}$$

$$j_n(x) = \frac{2e^2}{h}\int_{x_n}^{L} \mathrm{d}x'\, E(x'\mathrm{sgn}(x))\{\exp[\mathrm{i}\omega^*\tau_n(|x|,x')\mathrm{sgn}(|x|-x')] - \exp[\mathrm{i}\omega^*(\tau_n(|x|,x_n)+\tau_n(x',x_n))]\}\theta(|x|-x_n). \tag{27}$$

Here x_n is the absolute value of the critical (turning) point, which is determined by the condition

$$\varepsilon_n(x_n) = \mu. \tag{28}$$

From equations (23), (24) and (26), (27), it is apparent that the transport through a QPC is described by highly non-local (integral) operators. This suggests that the charge and current densities at a given point x are influenced by the electrical field within the whole conductor. Thus, the PWDF formalism allowed us to derive the charge and the current densities as non-local operators with respect to the electric field.

3. The admittance of the QPC

Our formulation of the kinetic equation for the PWDF allows us to describe the adiabatic transport through a QPC. Using equations (23), (24) and (26), (27), we can calculate the charge and current densities in the QPC, once the field distribution within the QPC is given. Of particular experimental interest is the calculation of the frequency dependence of the admittance of the QPC, the behaviour of which reveals more detailed information than any static characteristics.

It is well known that the static conductance is fully specified by the potential difference (bias voltage) between the right and left reservoirs, whereas the detailed electrical potential profile does not influence it significantly [1]. This result was derived using the Landauer formalism [10, 32], when the conductance was defined by the matrix of the transmission coefficients of the electrons corresponding to the different propagating channels. We can

readily show that this result also follows immediately from our PWDF approach. In the ballistic regime, when

$$L \ll l \tag{29}$$

($2L$ is the distance between the reservoirs, l is the mean free path), for $\omega, \nu \to 0$, we find for the propagating modes (open channels)

$$j_n = \frac{2e^2}{h} V \qquad V = \int_{-L}^{L} \mathrm{d}x\, E(x) \tag{30}$$

and for the non-propagating modes (closed channels)

$$j_n = 0. \tag{31}$$

Using equation (15), we obtain, for the total current flowing through the QBC, the result

$$I = \int_{-\infty}^{\infty} \mathrm{d}y\, j(y). \tag{32}$$

Hence the static conductance assumes the familiar form [1]

$$G = \frac{I}{V} = \frac{2e^2}{h}\mathcal{N} \tag{33}$$

where $\mathcal{N}$ is the number of open channels:

$$\mathcal{N} = \left[\frac{2k_F d(0)}{\pi}\right] \qquad \hbar k_F = \sqrt{2m\mu}. \tag{34}$$

Here the brackets $[\cdots]$ stand for the integral part of the enclosed expression. From these equations it is clear that the static conductance does not depend on the details of the smooth function, $d(x)$.

More generally, we can use the formalism of the PWDF to calculate the admittance at the frequency ω. From formulae (24)–(28) one can see that the partial current j_n is a function of the longitudinal coordinate x at $\omega \neq 0$. The continuity equation

$$\mathrm{div}\boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0 \tag{35}$$

in the QPC at $\omega \neq 0$ takes the form

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial x}\left\{ j_n - \mathrm{i}\omega \int_{-L}^{x} \mathrm{d}x'\, \rho_n(x') \right\} = \frac{\partial}{\partial x}\{I_{tot}\} = 0 \tag{36}$$

where

$$I_{tot} = \sum_{n=1}^{\infty} \left\{ j_n - \mathrm{i}\omega \int_{-L}^{x} \mathrm{d}x'\, \rho_n(x') \right\}.$$

Note that the total current I_{tot}, which includes the current density $\sum_{n=1}^{\infty} j_n(x)$ and displacement current $-\mathrm{i}\omega \sum_{n=1}^{\infty} \int_{-L}^{x} \mathrm{d}x'\, \rho_n(x')$ is independent of the longitudinal coordinate x. From the form of equation (36) it is easy to see that the displacement current vanishes within the left reservoir, so the total current is

$$I_{tot} = \sum_{n=1}^{\infty} j_n(-L) \tag{37}$$

and the admittance can be determined as

$$Y = \frac{I_{tot}}{V} = \frac{1}{V} \sum_{n=1}^{\infty} j_n(-L). \tag{38}$$

Thus, outside the QPC region, $|x| > L$, where $-L$ and L are the beginning and the end of the constriction ($-L < x < L$), the displacment current vanishes in the reservoirs; the total current is determined by equation (37), and the admittance is determined by equation (38).

In the general case, we need to determine the field $E(x)$ within the QPC from the Maxwell equations and afterwards calculate the admittance. Here we consider the long-wavelength approximation, in which

$$v_n^* \gg \omega L_n. \tag{39}$$

Here v_n^* is the typical velocity for the electrons of the nth channel and L_n characterizes the length of a region for each channel. For the open (propagating) modes L_n is the distance between the reservoirs ($L_n \sim 2L$) and

$$v_n^* = v_n(0). \tag{40}$$

For non-propagating modes (closed channels), L_n is twice the distance between the turning point (28) and the nearest reservoir ($L_n \sim 2(L - x_n)$). The typical velocity in this case is

$$v_n^* = \frac{2v_F}{\tilde{L}}\sqrt{x_n(L - x_n)} \qquad v_F = \sqrt{\frac{2\mu}{m}} \tag{41}$$

where $2\tilde{L}$ is the length of the constriction.

Condition (39) implies that the field changes only slightly during the time that it takes an electron to travel through the QPC. Hence equation (39) is the condition for weak frequency dispersion of the conductivity. To calculate the current of the propagating modes (open channels), we can approximate the velocity v_n as

$$v_n(x) \simeq v_n(0) = v_n^* \tag{42}$$

and for the reflecting modes

$$v_n(x) \simeq v_n^*\sqrt{\frac{|x| - x_n}{L - x_n}}. \tag{43}$$

We approximate the form of the QPC (as in [12]):

$$d(x) = d_0 \exp[(x/\tilde{L})^2]. \tag{44}$$

Using this approximation, we find for the open channels

$$j_n(-L) = \frac{2e^2}{h}\left(1 + \mathrm{i}\frac{\omega L}{v_n^*}\right)V \tag{45}$$

and for the closed channels

$$j_n(-L) = -\mathrm{i}\omega\frac{8e^2}{h}\frac{(L - x_n)}{v_n^*}\int_{x_n}^{L} \mathrm{d}x'\, E(x')\sqrt{\frac{x' - x_n}{L - x_n}}. \tag{46}$$

Consistent with our choice of a reflection symmetric $d(x)$, let us assume that the electric field inside the QPC is reflection symmetric, $E(x) = E(-x)$. In this case, the contribution of the open channels is determined by the total voltage V and is independent of the detailed profile of the electrical potential inside the QPC. Thus, we can write the admittance in the form

$$Y = G - \mathrm{i}\omega\mathcal{E} \tag{47}$$

where $G = (2e^2/h)\mathcal{N}$ is the static conductance. The emittance $\mathcal{E}$ of the QPC is given by the expression

$$\mathcal{E} = -G\frac{L}{\overline{v}^{(o)}} + \frac{16}{3}\frac{e^2}{h}\sum_{n=\mathcal{N}+1}^{\mathcal{N}+\tilde{\mathcal{N}}}\frac{\xi_n}{v_n^*}(L - x_n). \tag{48}$$

Here $\overline{v}^{(o)}$ is the harmonic mean of the velocities v_n^* in the open channels (42):

$$\frac{1}{\overline{v}^{(o)}} = \frac{1}{\mathcal{N}}\sum_{n=1}^{\mathcal{N}}\frac{1}{v_n^*}. \tag{49}$$

The integer $\tilde{\mathcal{N}}$ determines the number of closed channels:

$$\tilde{\mathcal{N}} = \left[\frac{2k_F d_0}{\pi}\exp[(L/\tilde{L})^2]\right] - \mathcal{N} \tag{50}$$

with $2L$ being the distance between the reservoirs. The discrete value ξ_n characterizes the relative bias of voltage in the region (x_n, L) filled with the electrons of the nth reflecting channel:

$$\xi_n = \frac{3}{2}\int_{x_n}^{L} \mathrm{d}x' \, \frac{E(x')}{V}\sqrt{\frac{x' - x_n}{L - x_n}}. \tag{51}$$

From equation (48) it follows immediately that the contribution of the reflecting modes to the emittance $\mathcal{E}$ is positive, whereas the contribution of the propagating modes is negative. This observation allows us to express our results concisely in terms of the equivalent circuit shown in figure 3. The admittance of the circuit is

$$Y = G - \mathrm{i}\omega(C - \Lambda G^2/c^2) \tag{52}$$

with

$$\omega C \ll G \qquad \omega\Lambda \ll c^2 G^{-1}.$$

The effective inductance in equation (52) is

$$\Lambda = \frac{c^2 L}{G\overline{v}^{(o)}} \tag{53}$$

and the effective capacitance is

$$C = \frac{16}{3}\frac{e^2}{h}\sum_{n=\mathcal{N}+1}^{\mathcal{N}+\tilde{\mathcal{N}}}\frac{\xi_n}{v_n^*}(L - x_n) \tag{54}$$

where ξ_n is defined in (51).

Note that equation (52) coincides with the general expression for the emittance derived in [18] (see equation (7) in [18]), where the emittance was expressed in terms of the geometric capacitance, transmission probability, and the densities of states of the 'mesoscopic capacitor plates'. Our description has allowed us to represent the emittance in terms of the inductance (53) and the capacitance (54), which are expressed in an explicit form through the microscopic characteristics, such as the harmonic mean of the velocities of the open channels (inductance), and the relative bias of voltage of the QPC ξ_n (51), velocities v_n^*, and the values of the turning points x_n (see (54)). It is easy to see that the capacitance (54) and the inductance (53) demonstrate a stepwise behaviour as functions of the gate voltage. This stepwise behaviour of the emittance, as was pointed out in [18], is in synchronism with the conductance steps, and is determined by the number of open (or closed) channels in the QPC.

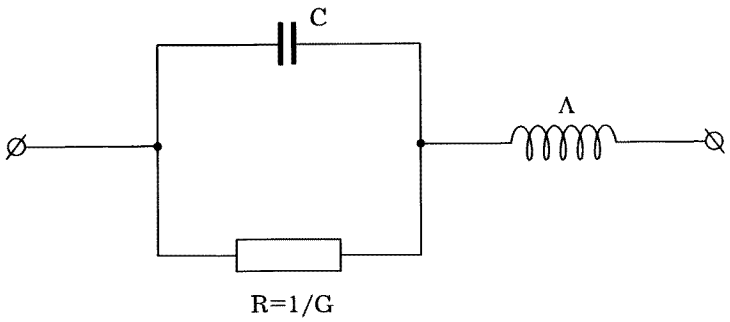

Figure 3. Equivalent circuit of the QPC.

We can readily show that the emittance is a stepwise function of the gate voltage. When the gate voltage approaches a point for which $2k_F d/\pi$ is integer, and one more mode opens (or closes), the inductance and the capacitance in expressions (53) and (54) increase to infinity.

In this case, condition (39) is violated, and the contribution of these points to the admittance must be calculated separately. Let us analyse the asymptotic behaviour of the emittance in this case. The approximation (42) and (43) is justified only if for all modes the parameter

$$\gamma_n = (\mu - \varepsilon_n(0))/\varepsilon_n(0)$$

is not too small. The situation when γ_n becomes small for the n_0th mode ($n_0 = \mathcal{N}, \mathcal{N}+1$) means that the corresponding mode is near to the point where it transforms from propagating to non-propagating, or *vice versa*. When $|\gamma_{\mathcal{N}}| \ll 1$ (for an open channel), we find that in inequality (39) and equation (49) the typical velocity for $n = \mathcal{N}$ is

$$v^*_{\mathcal{N}} \simeq v_F \frac{L}{\tilde{L}} \frac{\sqrt{2}}{\ln(4L^2/\tilde{L}^2|\gamma_{\mathcal{N}}|)}. \tag{55}$$

If $|\gamma_{\mathcal{N}+1}| \ll 1$ (for a closed channel), then in inequality (39) and in equations (48) and (53)

$$v^*_{\mathcal{N}+1} \simeq \frac{16}{3} v_F \frac{L}{\tilde{L}} \frac{\sqrt{2}}{\ln(4L^2/\tilde{L}^2|\gamma_{\mathcal{N}+1}|)}. \tag{56}$$

Hence the contribution of the $\mathcal{N}$th mode to the inductance is

$$\Lambda_{\mathcal{N}} \simeq \frac{c^2}{G^2} \frac{2e^2}{h} \frac{\tilde{L}}{v_F\sqrt{2}} \ln\left(\frac{4L^2}{|\gamma_{\mathcal{N}}|\tilde{L}^2}\right) \tag{57}$$

and the contribution of the $(\mathcal{N}+1)$th mode to the capacitance is

$$C_{\mathcal{N}+1} \simeq \frac{e^2}{h} \frac{\tilde{L}}{v_F\sqrt{2}} \ln\left(\frac{4L^2}{|\gamma_{\mathcal{N}+1}|\tilde{L}^2}\right). \tag{58}$$

If a channel opens (closes), and $\gamma_n \to 0$, there can be a case of strong frequency and spatial dispersion. Because of this, at these points the system cannot be treated in terms of effective inductance and capacitance. (Note, that in equations (55)–(58), when $\gamma_n \to 0$, the modules $|\gamma_{\mathcal{N}}|$ and $|\gamma_{\mathcal{N}+1}|$ should be substituted by $\sqrt{\gamma_n^2 + (L/l)^2}$, where l is the electron's mean free path in the ballistic quantum constriction, $L/l \ll 1$.)

The emittance is described by parameters of a different nature. The inductance Λ is determined by the velocities v_n for the open channels, and the capacitance C is mainly determined by the distribution of the electrical field as well as by the location of the turning points (28). The mesoscopic emittance can be controlled by the gate voltage.

4. Conclusions

We have developed a new approach, based on a partial Wigner distribution function, to analyse a.c. electron transport properties of a quantum point contact. Treating the quantum ballistic constriction in the adiabatic approximation, we have derived a Boltzmann-like equation for the partial Wigner distribution function in an effective potential brought about by the quantized transverse modes. We have analysed this equation in terms of propagating and reflecting trajectories in the quasiclassical approximation.

Our results establish that the a.c. electron transport depends directly on the number of propagating and reflecting modes, and that certain features are sensitive to the form of the distribution of the electric field in the QPC. In particular, the real part of the admittance (the conductance) is determined by the number of propagating electron modes, and does not depend on the spatial distribution of the electric field inside the QPC [1] . The imaginary part of the admittance (the emittance) exhibits stepwise oscillations as a function of the gate voltage and consists of two parts: the quantum inductance and the quantum capacitance. The quantum inductance is determined by the harmonic mean of the velocities for the propagating electron modes. The quantum mesoscopic capacitance is specified by the reflecting modes that are very sensitive to the geometry of the QPC. The emittance can be controlled by the gate voltage. Therefore, the measurements of the admittance can be more informative than the measurements of the static conductance.

It is important to stress that the effective quantum inductance and capacitance, and the equivalent circuit, are concepts valid within our linear-response, low-frequency approximation. For the high-frequency case, and when new propagating and non-propagating modes can appear or disappear, the frequency dispersion of the admittance is more complicated than the linear one given by the equivalent circuit of equation (52). This case must be considered using the self-consistent Maxwell equations for the electric field in the QPC. We are presently investigating this problem.

In connection with this remark we note that in the present paper we have considered the model of the QPC where along the y-direction (see figure 1) the walls are considered impenetrable (infinite potential barriers). Along the x-direction the potential changes adiabatically according to the adiabatical model (see [1]). In a real semiconductor 2DEG QPC the boundaries are created by the gate voltage. So, in this case the boundaries in the y-direction can also be 'smooth'. Analogously, both smooth and hard walls can also be realized in 3D quantum micro-constrictions of various shapes (see, for example, [33, 34], and references therein). In particular, the walls can be considered smooth in the presence of a magnetic field [33].

The WDF method suggested in this paper can easily be generalized to a description for both linear and nonlinear electron transport in 2D and 3D QPC and quantum wires taking into consideration the effects of gates, namely when the boundaries are smooth with an adiabatically varying shape. We are considering this problem at present.

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